Minimum Length from First Principles

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Abstract

We show that no device or gedanken experiment is capable of measuring a distance less than the Planck length. By "measuring a distance less than the Planck length" we mean, technically, resolve the eigenvalues of the position operator to within that accuracy. The only assumptions in our argument are causality, the uncertainty principle from quantum mechanics and a dynamical criteria for gravitational collapse from classical general relativity called the hoop conjecture. The inability of any gedanken

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experiment to measure a sub-Planckian distance suggests the existence of a minimal length.

In this work we show that quantum mechanics and classical general relativity considered simultaneously imply the existence of a minimal length, i.e. no operational procedure exists which can measure a distance less than this fundamental length. The key ingredients used to reach this conclusion are the uncertainty principle from quantum mechanics, and gravitational collapse from classical general relativity.

A dynamical condition for gravitational collapse is given by the hoop conjecture [1]: if an amount of energy E is confined at any instant to a ball of size R, where $R < E$, then that region will eventually evolve into a black hole[1].

From the hoop conjecture and the uncertainty principle, we immediately deduce the existence of a minimum ball of size l_P. Consider a particle of energy E which is not already a black hole. Its size r must satisfy

$$r \gtrsim \mathbf{max}\,[\,1/E\,,\,E\,] \quad , \tag{1}$$

where $\lambda_C \sim 1/E$ is its Compton wavelength and E arises from the hoop conjecture. Minimization with respect to E results in r of order unity in Planck units or $r \sim l_P$. If the particle *is* a black hole, then its radius grows with mass: $r \sim E \sim 1/\lambda_C$. This relationship suggests that an experiment designed (in the absence of gravity) to measure a short distance $l << l_P$ will (in the presence of gravity) only be sensitive to distances $1/l$.

Let us give a concrete model of minimum length. Let the position operator $\hat{x}$ have discrete eigenvalues $\{x_i\}$, with the separation between eigenvalues either of order l_P or smaller. (For regularly distributed eigenvalues with a constant separation, this would be equivalent to

[1] We use natural units where $\hbar, c$ and Newton's constant (or l_P) are unity. We also neglect numerical factors of order one.

a spatial lattice.) We do not mean to imply that nature implements minimum length in this particular fashion - most likely, the physical mechanism is more complicated, and may involve, for example, spacetime foam or strings. However, our concrete formulation lends itself to detailed analysis. We show below that this formulation cannot be excluded by any gedanken experiment, which is strong evidence for the existence of a minimum length.

Quantization of position does not by itself imply quantization of momentum. Conversely, a continuous spectrum of momentum does not imply a continuous spectrum of position. In a formulation of quantum mechanics on a regular spatial lattice, with spacing a and size L, the momentum operator has eigenvalues which are spaced by $1/L$. In the infinite volume limit the momentum operator can have continuous eigenvalues even if the spatial lattice spacing is kept fixed. This means that the displacement operator

$$\hat{x}(t) - \hat{x}(0) = \hat{p}(0)\frac{t}{M} \tag{2}$$

does not necessarily have discrete eigenvalues (the right hand side of (2) assumes free evolution; we use the Heisenberg picture throughout). Since the time evolution operator is unitary the eigenvalues of $\hat{x}(t)$ are the same as $\hat{x}(0)$. Importantly though, the spectrum of $\hat{x}(0)$ (or $\hat{x}(t)$) is completely unrelated to the spectrum of the $\hat{p}(0)$, even though they are related by (2). A measurement of arbitrarily small displacement (2) does not exclude our model of minimum length. To exclude it, one would have to measure a position eigenvalue x *and* a nearby eigenvalue x', with $|x - x'| << l_P$.

Many minimum length arguments are obviated by the simple observation of the minimum ball. However, the existence of a minimum ball does not by itself preclude the localization of a macroscopic object to very high precision. Hence, one might attempt to measure the

spectrum of $\hat{x}(0)$ through a time of flight experiment in which wavepackets of primitive probes are bounced off of well-localised macroscopic objects. Disregarding gravitational effects, the discrete spectrum of $\hat{x}(0)$ is in principle obtainable this way. But, detecting the discreteness of $\hat{x}(0)$ requires wavelengths comparable to the eigenvalue spacing. For eigenvalue spacing comparable or smaller than l_P, gravitational effects cannot be ignored, because the process produces minimal balls (black holes) of size l_P or larger. This suggests a direct measurement of the position spectrum to accuracy better than l_P is not possible. The failure here is due to the use of probes with very short wavelength.

A different class of instrument, the interferometer, is capable of measuring distances much smaller than the size of any of its sub-components. Nevertheless, the uncertainty principle and gravitational collapse prevent an arbitrarily accurate measurement of eigenvalue spacing. First, the limit from quantum mechanics. Consider the Heisenberg operators for position $\hat{x}(t)$ and momentum $\hat{p}(t)$ and recall the standard inequality

$$(\Delta A)^2(\Delta B)^2 \geq \ -\frac{1}{4}(\langle[\hat{A},\hat{B}]\rangle)^2 \quad . \tag{3}$$

Suppose that the position of a *free* test mass is measured at time $t = 0$ and *again* at a later time. The position operator at a later time t is

$$\hat{x}(t) = \hat{x}(0) \ + \ \hat{p}(0)\frac{t}{M} \quad . \tag{4}$$

We assume a free particle Hamiltonian here for simplicity, but the argument can be generalized [4]. The commutator between the position operators at $t = 0$ and t is

$$[\hat{x}(0), \hat{x}(t)] \ = \ i\frac{t}{M} \quad , \tag{5}$$

so using (3) we have

$$|\Delta x(0)||\Delta x(t)| \geq \frac{t}{2M} \quad . \tag{6}$$

We see that at least one of the uncertainties $\Delta x(0)$ or $\Delta x(t)$ must be larger than of order $\sqrt{t/M}$. As a measurement of the discreteness of $\hat{x}(0)$ requires *two* position measurements, it is limited by the greater of $\Delta x(0)$ or $\Delta x(t)$, that is, by $\sqrt{t/M}$,

$$\Delta x \equiv \mathbf{max}\left[\Delta x(0), \Delta x(t)\right] \geq \sqrt{\frac{t}{2M}} \quad , \tag{7}$$

where t is the time over which the measurement occurs and M the mass of the object whose position is measured. In order to push Δx below l_P, we take M to be large. Note that this is not the standard quantum limit [2] which can be overcome using refined techniques [3]. In order to avoid gravitational collapse, the size R of our measuring device must also grow such that $R > M$. However, by causality R cannot exceed t. Any component of the device a distance greater than t away cannot affect the measurement, hence we should not consider it part of the device. These considerations can be summarized in the inequalities

$$t > R > M \quad . \tag{8}$$

Combined with (7), they require $\Delta x > 1$ in Planck units, or

$$\Delta x > l_P \ . \tag{9}$$

Notice that the considerations leading to (7), (8) and (9) were in no way specific to an interferometer, and hence are *device independent*. We repeat: no device subject to quantum mechanics, gravity and causality can exclude the quantization of position on distances less than the Planck length.

It is important to emphasize that we are deducing a minimum length which is parametrically of order l_P, but may be larger or smaller by a numerical factor. This point is relevant to the question of whether an experimenter might be able to transmit the result of the measurement before the formation of a closed trapped surface, which prevents the escape of any signal. If we decrease the minimum length by a numerical factor, the inequality (7) requires $M >> R$, so we force the experimenter to work from deep inside an apparatus which has far exceeded the criteria for gravitational collapse (i.e., it is much denser than a black hole of the same size R as the apparatus). For such an apparatus a horizon will already exist before the measurement begins. The radius of the horizon, which is of order M, is very large compared to R, so that no signal can escape.

An implication of our result is that there may only be a finite number of degrees of freedom per unit volume in our universe - no true continuum of space or time. Equivalently, there is only a finite amount of information or entropy in any finite region our universe.

One of the main problems encountered in the quantization of gravity is a proliferation of divergences coming from short distance fluctuations of the metric (or graviton). However, these divergences might only be artifacts of perturbation theory: minimum length, which is itself a non-perturbative effect, might provide a cutoff which removes the infinities. This conjecture could be verified by lattice simulations of quantum gravity (for example, in the Euclidean path integral formulation), by checking to see if they yield finite results even in the continuum limit.

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